Gravitational-Wave Driven Instability of Rotating Relativistic Stars

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(Received)

A brief review of the stability of rotating relativistic stars is followed by a more detailed discussion of recent work on an instability of r-modes, modes of rotating stars that have axial parity in the slow-rotation limit. These modes may dominate the spin-down of neutron stars that are rapidly rotating at birth, and the gravitational waves they emit may be detectable.

§1. Introduction

A series of recent surprises appear dramatically to have improved the likelihood that the spin of rapidly rotating, newly formed neutron stars is limited by a nonaxisymmetric instability driven by gravitational waves – and that the emitted waves may be detectable.

The first of these was the discovery that the r-modes, rotationally restored modes that have axial parity for spherical models, are unstable in perfect fluid models with arbitrarily slow rotation. First indicated in numerical work by Andersson[1], the instability is implied in a nearly newtonian context by the newtonian expression for the r-mode frequency (3·1), and a computation by Friedman and Morsink[19] of the canonical energy of initial data showed (independent of assumptions on the existence of discrete modes) that the instability is a generic feature of axial-parity perturbations of relativistic stars.

Studies of the viscous and radiative timescales associated with the r-modes (Lindblom et al.[38], Owen et al.[43], Andersson et al.[2], Kokkotas and Stergioulas[29], Lindblom et al.[37]) have revealed a second surprising result: The growth time of r-modes driven by current-multipole gravitational radiation is significantly shorter than had been expected, so short, in fact, that the instability to gravitational radiation reaction easily dominates viscous damping in hot, newly formed neutron stars (see Fig. 1 below). As a result, a neutron star that is rapidly rotating at birth now appears likely to spin down by radiating most of its angular momentum in

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gravitational waves. (See, however, the caveats below.)

Nearly simultaneous with these theoretical surprises was the discovery by Marshall et.al. [41] of a fast (16ms) pulsar in a supernova remnant (N157B) in the Large Magellanic Cloud. Estimates of the initial period put it in the 6-9ms range, implying the existence of a class of neutron stars that are rapidly rotating at birth. Hence, the newly discovered instability appears to set the upper limit on the spin of the newly discovered class of neutron stars.

We discuss these developments and the uncertainties that accompany them below, in the context of a review of the gravitational-wave driven instability of rotating stars. The structure and stability of rotating relativistic stars has recently been reviewed in detail (Stergioulas [56], Friedman [16], [17], Friedman and Ipser [18]), and a general discussion of the small oscillations of relativistic stars may be found in a recent review article by Kokkotas [26] (see also Kokkotas and Schmidt [27]).

The excitement over the r-mode instability has generated a large literature in the past two years. (Andersson [1], Friedman and Morsink [19], Kojima [24], Lindblom et al. [38], Owen et al. [43], Andersson, Kokkotas and Schutz [2], Kokkotas and Stergioulas [29], Andersson, Kokkotas and Stergioulas [3], Madsen [40], Hiscock [21], Lindblom and Ipser [34], Bildsten [6], Levin [30], Ferrari et al. [14], Spruit [54], Brady and Creighton [7], Lockitch and Friedman [39], Lindblom et al. [37], Beyer and Kokkotas [5], Kojima and Hosonuma [25], Lindblom [32], Schneider et al. [50], Rezzolla et al. [47], Yoshida and Lee [64]) [Earlier studies of the axial-parity oscillations of models of the neutron star crust were reported by van Horn [61] and by Schumaker and Thorne [51]); and Chandrasekhar and Ferrari discussed the resonant scattering of axial wave modes [10] and the coupling between axial and polar modes induced by stellar rotation [9].] Although we mention many of the recent contributions listed here, our summary is too short to allow much detail.

§2.   Gravitational-Wave Driven Instability of Rotating Relativistic Stars

Only recently have the oscillation modes of rotating relativistic stars begun to be accessible to numerical study (see below). Early work on the perturbations of such stars focused mainly on the criteria for their stability, and led to the discovery that all rotating perfect fluid stars are subject to a nonaxisymmetric instability driven by gravitational radiation. The instability was found by Chandrasekhar [8] for the $l = m = 2$ polar mode of the uniform-density, uniformly rotating Maclaurin spheroids. Although this mode is unstable only for rapidly rotating models, by looking at the canonical energy of initial data with arbitrary values of m, Friedman and Schutz [20] and Friedman [15] showed that the instability is a generic feature of rotating perfect fluid stars, that even slowly rotating perfect-fluid models are formally unstable.

For a normal mode of the form $e^{i(\sigma t + m\varphi)}$ this nonaxisymmetric instability acts in the following manner: In a non-rotating star, gravitational radiation removes positive angular momentum from a forward moving mode and negative angular momentum from a backward moving mode, thereby damping all time-dependent, non-axisymmetric modes. In a star rotating sufficiently fast, however, a backward

moving mode can be dragged forward as seen by an inertial observer; and it will then radiate positive angular momentum. The angular momentum of the mode, however, remains negative, because the perturbed star has lower total angular momentum than the unperturbed star. As positive angular momentum is removed from a mode with negative angular momentum, the angular momentum of the mode becomes increasingly negative, implying that its amplitude increases: The mode is driven by gravitational radiation.

The conclusion, that a mode is unstable if it is prograde relative to infinity and retrograde relative to the star is equivalent to requiring that its frequency satisfies the condition,

$$\sigma(\sigma + m\Omega) < 0. \tag{2·1}$$

For the polar f- and p-modes, the frequency is large and approximately real. Condition (2·1) will be met only if $|m\Omega|$ is of order $|\sigma|$, so that for a given angular velocity the instability will set in first through modes with large m.

The instability spins a star down by allowing it to radiate away its angular momentum in gravitational waves. However, to determine whether this mechanism may be responsible for limiting the rotation rates of actual neutron stars, one must also consider the effects of viscous damping on the perturbations. Detweiler and Lindblom [13] suggested that viscosity would stabilize any mode whose growth time was longer than the viscous damping time, and this was confirmed by Lindblom and Hiscock [33]. Recent work has indicated that the gravitational-wave-driven instability can only limit the rotation rate of hot neutron stars, with temperatures above the superfluid transition point, $T \sim 10^9$K, but below the temperature at which bulk viscosity apparently damps all modes, $T \sim 10^{10}$K (see Fig. 1). (Ipser and Lindblom [22]; Lindblom [31] and Lindblom and Mendell [35]) Because of uncertainties in the temperature of the superfluid phase transition and in our understanding of the dominant mechanisms for effective viscosity, even this brief temperature window is not guaranteed.

The instability appears to arise in modes with both polar and axial parity, with the distinction defined as follows. The spherical symmetry of a nonrotating star implies that its perturbations can be divided into two classes, polar and axial, according to their behavior under parity. Where polar tensor fields on a 2-sphere can be constructed from the scalars Y_l^m and their gradients ∇Y_l^m (and the metric on a 2-sphere), axial fields involve the pseudo-vector $\hat{r} \times \nabla Y_l^m$, and their behavior under parity is opposite to that of Y_l^m. That is, axial perturbations of odd l are invariant under parity, and axial perturbations with even l change sign. Because a rotating star is also invariant under parity, its perturbations can also be classified according to their behavior under parity. Although l is well-defined only for modes of the spherical configuration, if a mode varies continuously along a sequence of equilibrium configurations that starts with a spherical star and continues along a path of increasing rotation, the mode can be called axial (polar) if it is axial (polar) for the spherical star.

It is useful to subdivide stellar perturbations according to the physics dominating their behaviour. This classification was first developed by Cowling [11] for the
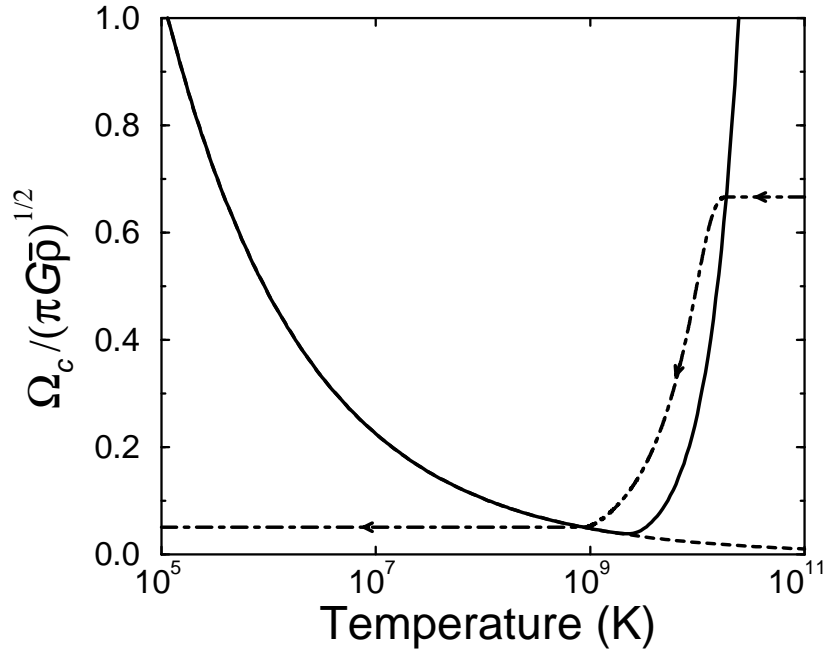
Fig. 1. Critical angular velocity vs. temperature for an $n = 1.0$ polytrope. Above the solid curve, the star rotates rapidly enough for its fastest growing ($l = m = 2$) r-mode to be unstable, whereas below the curve all modes are damped by viscosity. The dashed curve does not include the effects of bulk viscosity, while the solid curve does; however, for more accurate calulations of the bulk viscosity contribution see Lindblom, Mendell and Owen [37] and Andersson, Kokkotas and Schutz [2]. The dot-dashed curve shows the evolution of a rapidly rotating neutron star as it cools and spins down due to the emission of gravitational waves. [This figure is due to Lindblom, Owen and Morsink [38] and is reproduced here by permission of the authors.]

polar perturbations of newtonian polytropic models. The p-modes are polar-parity modes having pressure as their dominant restoring force [*] They typically have large pressure and density perturbations and high frequencies (higher than a few kilohertz for neutron stars). The other class of polar-parity modes are the g-modes, which are chiefly restored by gravity. They typically have very small pressure and density perturbations and low frequencies. Indeed, for isentropic stars, which are marginally stable to convection, the g-modes are all zero-frequency and have vanishing perturbed pressure and density. Similarly, all axial-parity perturbations of newtonian perfect fluid models have zero frequency in a non-rotating star. The perturbed pressure and density as well as the radial component of the fluid velocity are all rotational scalars and must have polar parity. Thus, the axial perturbations of a spherical star are simply stationary horizontal fluid currents.

The analogues of these modes in relativistic models of neutron stars have been studied by many authors. More recently, an additional class of outgoing modes has been identified that exist only in relativistic stars. Like the modes of black holes,

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[*] The lowest p-mode for each value of l and m is termed an f-mode or fundamental mode.

these are essentially associated with the dynamical spacetime geometry and have been termed w-modes, or gravitational wave modes. Their existence was first argued by Kokkotas and Schutz [28]. The polar w-modes were first found by Kojima [23] as rapidly damped modes of weakly relativistic models, while the axial w-modes were first studied by Chandrasekhar and Ferrari [10] as scattering resonances of highly relativistic models. (See the reviews by Kokkotas [26] and Kokkotas and Schmidt [27].) All w-modes appear to be stable, and if they are analogous to the modes of non-interacting zero-rest-mass test fields on the geometry of a rotating star, they should be unstable only when an ergosphere is present; but we are not aware of a careful study to rule out w-mode instability.

Until recently, the polar p-modes were expected to dominate the CFS instability through their coupling to mass multipole radiation. The relativistic stability points of p-modes have recently been found by Stergioulas and Friedman [55], [57] and, in the Cowling approximation, by Yoshida and Eriguchi [63]. This last reference also provides the first computation of the modes of rapidly rotating relativistic stars. The relativistic computation shows an instability of each at significantly smaller values of dimensionless measures of rotation than is seen in nearly newtonian stars. Particularly striking is the fact that the l=m=2 bar mode is unstable for relativistic polytropes of index n=1.0. The classical Newtonian result for the onset of the bar mode instability, $n < 0.808$) is replaced by $n < 1.3$ for relativistic polytropes.

Fig. 2 shows the critical angular velocity above which the lowest $l = m$ p-modes are unstable in perfect fluid models. For most realistic equations of state, Morsink, Stergioulas and Blattnig [42] find that the $l = m = 2$ bar mode is unstable in $1.4M_\odot$ neutron stars for angular velocities above $0.8\Omega_K$ to $0.95\Omega_K$.

The competition between viscosity and gravitational radiation is described by the dissipation equation,

$$\frac{1}{\tau} = \frac{1}{\tau_{\mathrm{GR}}} + \frac{1}{\tau_{\mathrm{shearviscosity}}} + \frac{1}{\tau_{\mathrm{bulkviscosity}}}, \tag{2·2}$$

with τ the e-folding time for each process. (The analysis sketched below can be found in Lindblom et al [38]; see also Ipser and Lindblom [22]). When the energy radiated per cycle is small compared to the energy of the mode, the imaginary part of the mode frequency is accurately approximated by the expression

$$\frac{1}{\tau} = -\frac{1}{2E}\frac{dE}{dt}, \tag{2·3}$$

where E is the energy of the mode as measured in the rotating frame,

$$E = \frac{1}{2} \int \left[\rho \delta v^a \delta v_a^* + \left(\frac{\delta p}{\rho} + \delta\Phi \right) \delta\rho^* \right] d^3x. \tag{2·4}$$

We have,

$$\frac{dE}{dt} = -\sigma(\sigma + m\Omega) \sum_{l \geq 2} N_l \sigma^{2l} \left(|\delta D_{lm}|^2 + |\delta J_{lm}|^2 \right)$$

$$- \int \left(2\eta \delta\sigma^{ab} \delta\sigma_{ab}^* + \zeta \delta\theta \delta\theta^* \right), \tag{2·5}$$
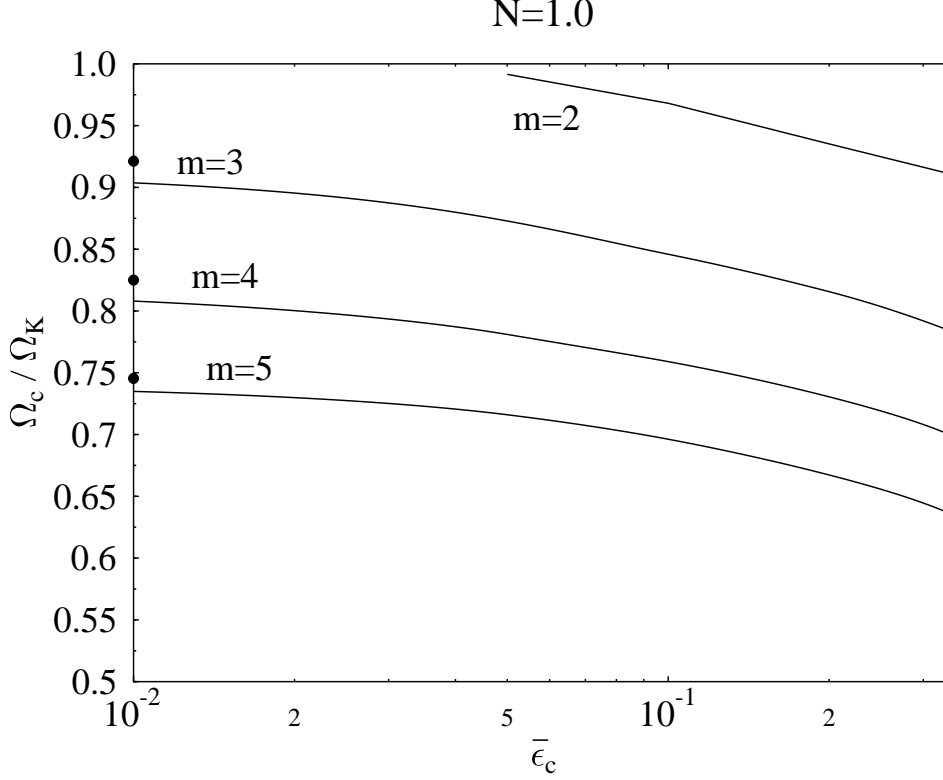
Fig. 2. Critical ratio of angular velocity to Kepler velocity vs. a dimensionless central energy density $\bar{\epsilon}_c$ for the $m = 2, 3, 4$ and 5 neutral modes of $n = 1.0$ polytropes. The largest value of $\bar{\epsilon}_c$ shown corresponds to the most relativistic stable configurations, while the lowest $\bar{\epsilon}_c$ corresponds to less relativistic configurations. The filled circles on the vertical axis represent the Newtonian limit.

where the dissipation due to gravitational radiation [58] has coupling constant

$$N_l = \frac{4\pi G}{c^{2l+1}} \frac{(l+1)(l+2)}{l(l-1)[(2l+1)!!]^2}; \qquad (2{\cdot}6)$$

$\delta\sigma_{ab}$ and $\delta\theta$ are the coefficients of shear and bulk viscosity; and the corresponding coefficients η and ζ are estimated (Cutler and Lindblom [12]; Sawyer [49]) by

$$\eta = 2 \times 10^{18} \left(\frac{\rho}{10^{15}\,\mathrm{g\cdot cm^{-3}}}\right)^{\frac{9}{4}} \left(\frac{10^9 K}{T}\right)^2 \ \mathrm{g\cdot cm^{-1}\cdot s^{-1}}, \qquad (2{\cdot}7)$$

and

$$\zeta = 6 \times 10^{25} \left(\frac{1\mathrm{Hz}}{\sigma + m\Omega}\right)^2 \left(\frac{\rho}{10^{15}\,\mathrm{g\cdot cm^{-3}}}\right)^2 \left(\frac{T}{10^9 K}\right)^6 \ \mathrm{g\cdot cm^{-1}\cdot s^{-1}}; \qquad (2{\cdot}8)$$

Polar and axial radiation arise, respectively, from mass and current multipoles, D_{lm} and J_{lm}, given by the equations,

$$D_{lm} = \int dV\, r^l \rho Y_{lm}^* \qquad\qquad J_{lm} = \int dV\, r^l \rho \mathbf{v} \cdot \mathbf{r} \times \nabla Y_{lm}^* \qquad (2\cdot 9)$$

The additional factor of v in the current multipoles implies an additional factor of v^2 in the radiated energy of axial modes, and hence a smaller expected rate of radiation for the same multipole: For a mode of amplitude $A =$ (displacement of fluid element)$/R$, with R the stellar radius, we have

$$\frac{dE}{dt} \sim A^2 M^2 R^{2l} \sigma^{2l+2} \qquad\qquad \frac{dE}{dt} \sim A^2 M^2 R^{2l+2} \sigma^{2l+4}. \qquad (2\cdot 10)$$

The extra factor of σ^2 and the fact that σ is proportional to Ω for slowly rotating stars led to the incorrect expectation that r-modes could be neglected.

§3. The r-mode Instability and its Implications

The unexpected apparent dominance of the r-modes arises from the fact that, where the polar $l = 2$ mode is unstable, if at all, only for $\Omega \sim \Omega_K$, the $l = m = 2$ r-mode is unstable (for perfect-fluid models) for arbitrarily small Ω,[1), 19)] The instability in models of slowly rotating, nearly newtonian stars, follows from the fact that the frequencies,[44)]

$$\sigma + m\Omega = \frac{2m\Omega}{l(l+1)}. \qquad (3\cdot 1)$$

satisfy the criterion (2·1),

$$\sigma(\sigma + m\Omega) = -\frac{2(l-1)(l+2)m^2\Omega^2}{l^2(l+1)^2} < 0. \qquad (3\cdot 2)$$

[A computation by Friedman and Morsink[19)] of the canonical energy of initial data showed (independent of assumptions on the existence of discrete modes) that the instability is a generic feature of axial-parity fluid perturbations of relativistic stars.]

Not only are p-modes with low values of m unstable only for large values of Ω, their frequencies are near zero, because the frequency of a mode vanishes at its instability point, and a star cannot rotate fast enough for the lowest p-modes to be far from their instability points. Thus where p-modes may limit the rotation of a newly formed star to $0.9\Omega_K$,[35) *)] the r-modes are likely to limit rotation to less than $0.2\Omega_K$. Here Ω_K is the maximum angular velocity of a uniformly rotating star, equal to the angular velocity of a satellite in orbit at the star's equator.

The current picture that has emerged of the spin-down of a hot, newly formed neutron star can be readily understood in terms of a model of the r-mode instability due to Owen, Lindblom, Cutler, Schutz, Vecchio and Andersson (hereafter OLCSVA) [43)]. Since one particular mode (with spherical harmonic indices $l = m = 2$ and

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*) this is likely to decrease, perhaps to $0.8\ \Omega_K$ when the correct general-relativistic computation with viscosity is completed.

frequency $\sigma = -4\Omega/3$) is found to dominate the r-mode instability, the perturbed star is treated as a simple system with two degrees of freedom: the uniform angular velocity Ω of the equilibrium star, and the (dimensionless) amplitude α of the $l = m = 2$ r-mode. Initially, the neutron star forms with a temperature large enough for bulk viscosity to damp any unstable modes, $T \gtrsim 10^{10}$K; the star is assumed to be rotating close to its maximum (Kepler) velocity, $\Omega_K \sim \sqrt{M/R^3}$. The star then cools by neutrino emission at a rate given by a standard power law cooling formula (Shapiro and Teukolsky [52)]). Once it reaches the temperature window at which the $l = m = 2$ r-mode can go unstable, the system is assumed to evolve in three stages.

First, the amplitude of the r-mode undergoes rapid exponential growth from some arbitrary tiny magnitude. Using conservation of energy, (2·5) and the related equation for the rate of angular momentum loss to gravitational waves,

$$\frac{dJ}{dt} = -\frac{c^3}{16\pi G}\left(\frac{4\Omega}{3}\right)^5 J_{22}^2, \qquad (3\cdot3)$$

with both E and J proportional to α^2, OLCSVA derive the following equations for the evolution of the system in this stage. After expressing E and J in terms of Ω and α

$$\frac{d\Omega}{dt} = -\frac{2\Omega}{\tau_V}\frac{\alpha^2 Q}{1 + \alpha^2 Q} \qquad (3\cdot4)$$

$$\frac{d\alpha}{dt} = \frac{\alpha}{|\tau_{GR}|} - \frac{\alpha}{\tau_V}\frac{1 - \alpha^2 Q}{1 + \alpha^2 Q} \qquad (3\cdot5)$$

Here, τ_{GR} and τ_V are, respectively, the timescales for the growth of the mode by gravitational radiation reaction and the damping of the mode by viscosity (see Sect. 2.6). (The parameter Q is a constant of order 0.1 related to the initial angular momentum and moment of inertia of the equilibrium star.) Since the initial amplitude α of the mode is so small, the angular momentum changes very little at first (Eq. (3·4)). That this stage is characterized by the rapid exponential growth of α is the statement that the first term in Eq. (3·5) (the radiation reaction term) dominates over the second (viscous damping).

Eventually the mode will grow to a size at which linear perturbation theory is insufficient to describe its behavior. It is expected that a non-linear saturation will occur, halting the growth of the mode at some amplitude of order unity, although the details of these non-linear effects are poorly understood at present. When this saturation occurs, the system enters a second evolutionary stage during which the mode amplitude remains essentially unchanged and the angular momentum of the star is radiated away. During this stage OLCSVA evolve their model system according to the equations

$$\alpha^2 = \kappa \qquad (3\cdot6)$$

$$\frac{d\Omega}{dt} = -\frac{2\Omega}{|\tau_{GR}|}\frac{\kappa Q}{1 - \kappa Q} \qquad (3\cdot7)$$

where κ is constant of order unity parameterizing the uncertainty in the degree of non-linear saturation. The star spins down by Eq. (3·7), radiating away most of its angular momentum while continuing to cool gradually.

When its temperature and angular velocity are low enough that viscosity again dominates the gravitational-wave-driven instability, the mode will be damped. During this third stage, OLCSVA return to Eqs. (3·4)-(3·5) to continue the evolution of their system. That the mode amplitude decays is the statement that the second term in Eq. (3·5) (the viscous damping term) dominates the first (radiation reaction), at this temperature and angular velocity.

This three-stage evolutionary process leaves the newly formed neutron star with an angular velocity small compared with Ω_K. This final angular velocity appears to be fairly insensitive to the initial amplitude of the mode and to its degree of non-linear saturation. A final period $P \gtrsim 5 - 10$ms apparently rules out accretion-induced collapse of white dwarfs as a mechanism for the formation of millisecond pulsars with $P \lesssim 3$ms.

The r-mode instability has also revived interest in the Wagoner [62] mechanism, in which old neutron stars are spun up by accretion until the angular momentum loss in gravitational radiation balances the accretion torque. Bildsten [6] and Andersson, Kokkotas and Stergioulas [3] have proposed that the r-mode instability might succeed in this regard where the instability to polar modes seems to fail. However, the mechanism appears to be highly sensitive to the temperature dependence of viscous damping. Levin [30] has argued that if the r-mode damping is a decreasing function of temperature (at the temperatures expected for accreting neutron stars, $T \sim 10^8$K) then viscous reheating of the unstable neutron star could drive the system away from the Wagoner equilibrium state. Instead, the star would follow the cyclic evolution pattern depicted in Fig. 3. Initially, the runaway reheating would drive the star further into the r-mode instability regime (B-C in Fig. 3) and spin it down to a fraction of its angular velocity (C-D). Once it has slowed to the point at which the r-modes become damped, it would again slowly cool (D-A) and begin to spin up by accretion (A-B). Eventually, it would again reach the critical angular velocity for the onset of instability and repeat the cycle. Since the radiation spin-down time is of order 1 year, while the accretion spin-up time is of order 10^6 years, the star spends only a small fraction of the cycle emitting gravitational waves via the unstable r-modes. This would significantly reduce the likelihood that detectable gravitational radiation is produced by such sources. On the other hand, if the r-mode damping is independent of - or increases with - temperature (at $T \sim 10^8$K) then the Wagoner equilibrium state may be allowed (Levin [30]). Work is currently in progress (Lindblom and Mendell [36]) to investigate the r-mode damping by mutual friction in superfluid neutron stars, which was the dominant viscous mechanism responsible for ruling out the Wagoner scenario in the first place (Lindblom and Mendell [35]).

Other uncertainties in the scenarios described above are still to be investigated. There is substantial uncertainty in the cooling rate of neutron stars, with rapid cooling expected if stars have a quark interior or core, or a kaon or pion condensate. Madsen [40] suggests that an observation of a young neutron star with a rotation period below $5 - 10$ms would be evidence for a quark interior; but even without rapid cooling, the uncertainty in the superfluid transition temperature may allow a superfluid to form at about 10^{10}K, possibly killing the instability.

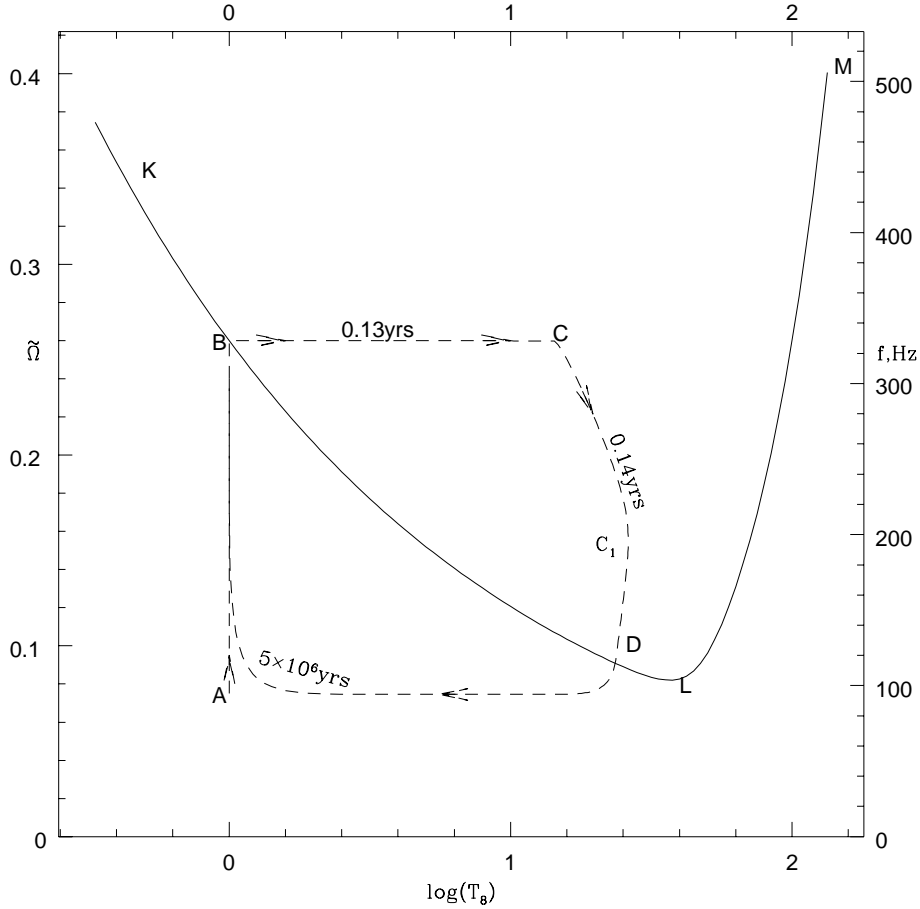We noted above the expectation that the growth of the unstable r-modes will

Fig. 3. Levin's [30] cyclic evolution scenario for an accreting neutron star when r-mode damping is a decreasing function of temperature. The solid curve K-L-M shows the critical angular velocity vs. temperature, while the dashed curve A-B-C-D-A shows the evolution of the star. [This figure is due to Levin [30] and is reproduced here by permission of the author.]

saturate at an amplitude of order unity due to non-linear effects (such as mode-mode couplings); however, this limiting amplitude is not yet known with any certainty and could be much smaller. In particular, it has been suggested that the non-linear evolution of the r-modes will wind up the magnetic field of a neutron star, draining energy away from the mode and eventually suppressing the unstable modes entirely (Rezzolla, Lamb and Shapiro [46]; see also Spruit [54]).

If the r-modes saturate near unity, a study by Brady and Creighton [7] following the work by Owen et al [43], finds that newly formed neutron stars should be detectable by LIGO II with narrow banding out to about 8 Mpc, with uncertainty allowing a range of perhaps 4-20 Mpc. The Virgo cluster is then likely to be out of reach, and even with this optimistic assumption about the saturation amplitude, r-modes are most likely not to be detected until sensitivity passes the limits of ad-

vanced LIGO.

Despite the recent intense interest in r-modes, they are not yet well-understood for stellar models appropriate to neutron stars. A neutron star is accurately described by a perfect fluid model in which both the equilibrium and perturbed configurations obey the same one-parameter equation of state. We call such models isentropic, because isentropic models and their adiabatic perturbations obey the same one-parameter equation of state.

For stars with more general equations of state, the r-modes describe the dynamical evolution of initial perturbations that have axial parity. This is not, however, the case for isentropic models. Early work on the r-modes focused on newtonian models with general equations of state (Papalouizou and Pringle[44], Provost et al.[45], Saio[48], Smeyers and Martens[53]) and mentioned only in passing the isentropic case. In isentropic newtonian stars, one finds that the only purely axial modes allowed are the r-modes with $l = m$ and simplest radial behavior. (Provost et al.[45]*)) It is these r-modes only that have been studied (and found to be physically interesting) in connection with the gravitational-wave driven instability.

The disappearance of the purely axial modes with $l > m$ occurs for the following reason.(Lockitch and Friedman[39]) Axial perturbations of a spherical star are time-independent convective currents with vanishing perturbed pressure and density. In spherical *isentropic* stars, stars for which both star and perturbation are governed by a single one-parameter equation of state, the gravitational restoring forces that give rise to the g-modes vanish and they, too, become time-independent convective currents with vanishing perturbed pressure and density. Thus, the space of zero frequency modes, which generally consists only of the axial r-modes, expands for spherical isentropic stars to include the polar g-modes. This large degenerate subspace of zero-frequency modes is split by rotation to zeroth order in the star's angular velocity, and the corresponding modes of rotating isentropic stars are generically hybrids whose spherical limits are mixtures of axial and polar perturbations.

These hybrid rotational modes have already been found analytically for the uniform-density Maclaurin spheroids by Lindblom and Ipser[34] and numerically for slowly rotating $n = 1$ polytropes by Lockitch and Friedman[39]; and Yoshida and Lee[64] have obtained these Newtonian polytropic modes to the next nonvanishing order in Ω. Because the restoring force for these modes is rotation, as it is for the axial r-modes, Ipser and Lindblom[34] call them rotational modes or generalized r-modes.

The r-modes of rotating relativistic stars were studied for the first time only recently (Andersson[1]; Kojima[24]; Beyer and Kokkotas[5]; Kojima and Hosonuma[25]). As in the newtonian case, a spherical isentropic relativistic star has a large degenerate subspace of zero-frequency modes consisting of the axial-parity r-modes[60] and the polar-parity g-modes.[59] Although isentropic newtonian stars retain a vestigial set of purely axial modes (those having $l = m$), rotating relativistic stars of this type have *no* pure r-modes[4], no modes whose limit for a spherical star is

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*) An appendix in this paper incorrectly claims that no $l = m$ r-modes exist, based on an incorrect assumption about their radial behavior.

purely axial. Instead, the newtonian r-modes with $l = m \geq 2$ acquire relativistic corrections with both axial and polar parity to become discrete hybrid modes of the corresponding relativistic models.

In the slow-motion approximation in which they have so far been studied, $r -$ *modes* of nonisentropic stars have, remarkably, a *continuous* spectrum. Kojima shows that the axial modes are described by a single, second-order ODE [24] for the modes' radial behavior; he argues that the continuous spectrum is implied by the vanishing of the coefficient of the highest derivative term of this equation at some value of the radial coordinate, and Beyer and Kokkotas [5] make the claim precise. As the latter authors point out, the continuous spectrum they find may be an artifact of the vanishing of the imaginary part of the frequency in the slow rotation limit. (Or, more broadly, it may be an artifact of the slow rotation approximation.)

In addition, Kojima and Hosonuma [25] have studied the mixing of axial and polar perturbations to order Ω^2 in rotating relativistic stars, again finding a continuous mode spectrum. Their calculation uses the Cowling approximation (which ignores all metric perturbations) and assumes an ordering of the perturbation variables in powers of Ω which forbids the mixing of axial and polar terms at zeroth order. Again the continuous spectrum may be an artifact of an approximation that enforces a purely real frequency.

Finally, several authors have looked at the gravitational-wave driven r-mode instability in white dwarfs. Andersson et al [3] suggested that the instability may limit the spin of accreting dwarfs; and Hiscock [21] found that an r-mode instability of dwarfs conforming to the Wagoner scenario – radiating in gravitational waves the angular momentum it gains in accretion – was potentially detectable by space-based detectors, with a preference for a shorter baseline instrument like LISA. Lindblom [32], however, finds that the growth time of the instability is too slow to allow the amplitude to reach saturation in less than 6×10^9 yr; and he finds a very restricted range of mass and accretion rates for which the star could be hot enough for long enough that the r-mode could grow to a dynamically significant amplitude. Although Lindblom's analysis allows instability in accreting dwarfs larger than $0.9\ M_\odot$ that are spun up by accretion to about $0.85\Omega_K$, observable consequences of the instability do not seem likely.

Summary

In a newborn, rapidly rotating neutron star, an r-mode instability appears to grow
  until it is in the nonlinear regime. It then steadily radiates the angular momentum
  of the star until the temperature falls below the superfluid transition temperature
  of about $10^9\ K$. By this time, the star's angular velocity will have decreased to
  less than 1/5 the maximum (Keplerian) Ω.

The formation of millisecond pulsars from accretion-induced collapse of white dwarfs
  is apparently ruled out for periods less than about 3 ms.

The observation of Marshall et al apparently implies that ordinary supernovae pro-
  duce neutron stars rotating fast enough to have unstable axial modes.

LIGO II may be able to detect the gravitational radiation emitted by this instability in supernovae within 4 to 20 Mpc.

All of these conclusions are based on linear perturbation theory with no magnetic fields, and work is in progress to decide whether the maximum mode amplitude is large enough to justify the dramatic predictions.

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